Int. J. Solids Structures, 1971, Vol. 7, pp. 161 to 181. Pergamon Press. Printed in Great Britain

REFLECTION OF OBLIQUE SHOCK WAVES IN ELASTIC SOLIDS

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Abstract—The reflection of a finite elastic plane shock wave at a plane boundary is examined. A semi-inverse method of solution is used. Only angles of incidence which are less than a critical angle are considered in detail. In general there are three reflected waves which may be either simple waves or shock waves. It is found that either kind of reflected wave may be completely specified by a single parameter provided conditions ahead of the wave are known. Boundary conditions at the plane surface must be solved for the specifying parameters. A simple existence theorem is stated, critical angle cases are discussed, and several simple examples are presented.

1. INTRODUCTION

MANY recent papers have been devoted to the search for solutions of finite amplitude wave problems in solids. These have been principally of two types. On the one hand, several authors have solved initial-boundary value problems in one spatial dimension.† On the other hand several authors have treated the conditions for existence of singular surfaces and for growth or decay at quite arbitrary acceleration fronts.‡ Such solutions are of great use in understanding and predicting major aspects of the response of elastic materials to finite excitations. The solution presented here is offered in the same spirit. That is to say, it should be regarded as a general case which may be used to describe important aspects of more detailed boundary–initial value problems.

Specifically the problem to be treated is the reflection of a finite plane shock wave from a plane surface where the angle between the shock wave and the surface is in the range $0 \leq \theta < \theta_c$. The maximum angle, θ_c, for which the simple reflection solutions hold is a critical angle and cannot be given in general, but must be determined in each special case. The exact value will depend on such factors as material constitution, incident shock strength, conditions ahead of the incident shock, and surface boundary conditions. Critical angle phenomena will be described qualitatively, but are not the primary focus here.

A strictly mechanical theory is used. Specifically, the material is assumed to be homogeneous, simple and elastic [9]. Thus the stress is a function solely of the deformation gradient taken with respect to a fixed reference configuration which is assumed to be stress free. Since no thermal effects are included, criteria other than entropy increase must be introduced to determine shock wave stability.

The reflected waves are found by a semi-inverse method. The incident shock is assumed to be given *a priori* so that $\mathscr{L}$, the line of intersection with the boundary, moves normal to itself at a known fixed speed. The basic assumption is then made that the reflection consists of families of simple waves [10], each wavelet of which passes through $\mathscr{L}$. In general three

† References [1–5] are representative.
‡ References [6–8] are representative.

simple waves are required. The problem then reduces to the determination of the distribution and strength of the wavelets by means of ordinary differential equations. In some cases a simple wave must be replaced by a shock wave.

In any case the solutions are of the two dimensional steady state type. They are two dimensional in that there is no dependence on a coordinate with axis parallel to $\mathscr{L}$. They are steady state in that an observer moving parallel to the surface and at the same speed as $\mathscr{L}$ will see only fixed deformation fields. Solutions of this type may be expected to have validity locally or asymptotically for curved shock reflections or they may hold exactly in finite regions bounded by other discontinuity surfaces.

In Section 2 a summary of the necessary finite elasticity theory and shock wave theory is given. The geometry of normal and wave cones behind the incident shock is discussed in Section 3. These two sections contain standard material in the main and are included to make the paper relatively self contained. References to complete expositions are given at the beginning of each subsection where possible.

The first two subsections of Section 4 contain the main substance of the paper. There the general solution for reflected simple waves is presented. The remainder of Section 4 contains comments on existence, reflected shock waves, and critical angles. Several special cases are discussed in Section 5.

2. ELASTIC SIMPLE WAVES AND SHOCK WAVES

Finite elasticity [9]

Let (X^1, X^2, X^3) and (x^1, x^2, x^3) be the Cartesian coordinates of an elastic particle. Here X^α are the coordinates of the particle in an unstressed reference state and x^i are the coordinates in the present configuration. The two coordinate sets are related by a non-singular one parameter family of mappings $x^i = x^i(X^\alpha, t)$ where the parameter t is time. The deformation gradient is denoted by $F^i_\alpha = x^i_{,\alpha} = \partial x^i/\partial X^\alpha$ and the velocity is denoted by $u^i = \dot{x}^i = \partial x^i/\partial t$. We have $\det \mathbf{F} \neq 0$.†

If the stress and velocity fields are differentiable, then the equations expressing balance of momentum, and moment of momentum respectively are:

$$\begin{aligned} T^{i\alpha}_{,\alpha} &= \rho_0 \dot{u}^i \\ F^j_\alpha T^{i\alpha} &= F^i_\alpha T^{j\alpha}. \end{aligned} \tag{1}$$

Here ρ_0 is the mass density in the reference configuration and $\mathbf{T}$ is the Piola stress tensor or engineering stress (i.e. force per unit reference area). The Piola stress tensor is related to the Cauchy stress tensor $\mathbf{t}$ by the formula $\mathbf{T} = (\det \mathbf{F})\,\mathbf{t}\,(\mathbf{F}^{-1})^T$ where the superscript T denotes transposition.

If the functions $x^i(X^\alpha, t)$ are continuous everywhere, but have discontinuous first derivatives on some propagating surface, Σ, then the differential balance conditions must be replaced by jump conditions on Σ. That which corresponds to equation (1_1) is:

$$[T^{i\alpha}]N_\alpha = -\rho_0 V [u^i]. \tag{2}$$

† Direct notation as well as index notation will be used when convenient. Thus the tensor $\mathbf{F}$ has Cartesian components F^i_α, the vector $\mathbf{u}$ has Cartesian components u^i, etc.

Moment of momentum adds nothing new, but kinematic jump conditions must be adjoined

$$[x^i] = 0$$
$$[F^i_\alpha] = a^i N_\alpha \tag{3}$$
$$[u^i] = -a^i V.$$

In these equations **N** is the unit normal to Σ in the reference configuration and points in the direction of propagation. The speed of propagation along the normal is V and **a** is the amplitude vector of the jump. The square brackets indicate the jump in the quantity enclosed taken across Σ; thus $[K] = K^+ - K^-$ where the plus and minus signs indicate the limit values at a point on the surface taken from the side of positive and negative normal respectively. Jump conditions for acceleration waves or other higher order singular surfaces could also be written, but will not be needed in the sequel.

The constitutive equations are assumed to be of the form

$$\mathbf{T} = \mathbf{T}(\mathbf{F}) \tag{4}$$

where the response function satisfies (1_2) identically and may be subject to restrictions of material symmetry as well as the principle of material indifference.

Equations (1)–(4) together with appropriate initial and boundary conditions form a strictly mechanical theory of simple elastic materials.

Simple waves [10]

Simple waves are defined to be regions of space-time in which the deformation gradient and velocity fields are continuous and depend on a single parameter, say $\gamma = G(\mathbf{X}, t)$. Regions of constant γ are propagating surfaces with unit normal and normal velocity in the reference configuration given by:

$$\mathbf{N}(\gamma) = \frac{\nabla G}{|\nabla G|}, \qquad V(\gamma) = -\frac{\dot{G}}{|\nabla G|}. \tag{5}$$

The equation of motion (1_2) and a compatibility equation may be written:

$$C^{\alpha\beta}_{ij} F'^j_\beta G_{,\alpha} = \rho_0 u'_i \dot{G}$$
$$F'^j_\beta \dot{G} = u'^j G_{,\beta} \tag{6}$$

where the prime indicates differentiation with respect to the parameter γ. The elasticities $C^{\alpha\beta}_{ij}$ are functions of **F** and are given by $C^{\alpha\beta}_{ij} = \partial T^\alpha_i / \partial F^j_\beta$. If $\dot{G} \neq 0$ equation (6) may be rewritten with the aid of (5) as

$$(C^{\alpha\beta}_{ij} N_\alpha N_\beta - \rho_0 V^2 \delta_{ij}) u'^j = 0$$
$$V F'^j_\beta + u'^j N_\beta = 0. \tag{7}$$

Furthermore, Varley shows that simple waves are regions swept out by one parameter families of propagating planes or wavelets

$$L(\gamma) = N_\alpha(\gamma) X^\alpha - V(\gamma) t. \tag{8}$$

Inversion of equation (8) yields the function $G(\mathbf{X}, t)$.

Equation (7_1) shows that simple waves are carried by characteristic surfaces and propagate with one of the characteristic speeds. It will be assumed for arbitrary direction of the normal vector that the acoustic tensor $Q_{ij} = C_{ij}^{\alpha\beta}N_\alpha N_\beta$ is positive definite for all $\mathbf{N}$ and $\mathbf{F}$ and has three linearly independent right proper vectors. This requirement assures that the propagation speeds be real and non-zero and that three distinct wave types exist. Furthermore, it will be generally assumed that the propagation speeds are distinct. This last requirement is very strong and must severely restrict the class of materials or deformations to be considered since isotropic materials, cubic materials, hexagonal materials, and perhaps others as well all exhibit some degeneracy of wave speeds for the case $\mathbf{F} = \mathbf{1}$. Fortunately, in some important special cases the requirements for three distinct speeds may be lifted, but in the general case the effect of degeneracy remains an open question.

Shock waves

Shock waves are propagating surfaces across which velocity and deformation gradients are discontinuous. Their motion is described by equations (2)–(4). If $\mathbf{F}^+$ and $\mathbf{u}^+$ are known, then (2) may be written

$$\{T^{i\alpha}(F^j_{+\beta}) - T^{i\alpha}(F^j_{+\beta} - a^jN_\beta)\}N_\alpha = \rho_0 V^2 a^i. \tag{9}$$

For a fixed direction of propagation, $\mathbf{N}$, and a fixed deformation gradient and velocity $\mathbf{F}_+, \mathbf{u}_+$ ahead of the wave, equation (9) gives three relations among the four quantities a^1, a^2, a^3 and V. If it may be solved for $\mathbf{a}$ in terms of V, then this solution, together with (3_2) and (3_3) rewritten as

$$\begin{aligned} F^i_{-\alpha} &= F^i_{+\alpha} - a^i(V)N_\alpha \\ u^i_- &= u^i_+ + Va^i(V) \end{aligned} \tag{3'}$$

determines a one parameter family of shock waves. Other parameterizations may of course be possible.

Let the three equations (9) be represented by the relations $K_i(\mathbf{a}, V) = 0$. Then the implicit function theorem [11] states that if (1) the K_i are continuously differentiable, (2) $\mathbf{a}_0$ and V_0 are such that $K_i(\mathbf{a}_0, V_0) = 0$, and (3) the determinant det $(\partial K_i/\partial a^j)_{\mathbf{a}_0,V_0} \neq 0$, then there is a neighborhood about $(\mathbf{a}_0, V_0)$ in which $K_i = 0$ may be solved uniquely for $\mathbf{a} = \mathbf{a}(V)$. One solution of (9) is given by $\mathbf{a} = \mathbf{0}$, $V = V_0$ where V_0 is an arbitrary positive number. The determinant in this case is given by

$$\det\left|\frac{\partial K_i}{\partial a_j}\right| = \det|C_{ij}^{\alpha\beta}(\mathbf{F}_+)N_\alpha N_\beta - \rho_0 V_0^2\delta_{ij}| \tag{10}$$

which of course is the characteristic determinant. If V_0 is not a characteristic velocity, then the determinant is not zero and the unique but trivial solution $\mathbf{a} \equiv \mathbf{0}$ holds in a neighborhood of $(\mathbf{a}_0, V_0)$.

Of greater interest is the case in which V_0 is a characteristic velocity, $V_0 = V^{(m)}$, $m = 1, 2, 3$. In this case the determinant is zero and the implicit function theorem does not apply. Let it be assumed that inversion is still possible, but rather than V as a parameter, let $(\mathbf{a}, V)$ be functions of the amplitude $a = \pm|\mathbf{a}|$. (The sign is set automatically as shown below.) If the functions are differentiable, then the derivatives $d^n\mathbf{a}/da^n$, d^nV/da^n evaluated at the point $(\mathbf{a}_0, V_0)$ may be found by repeated differentiation of equation (9) and the equation that defines a (except for sign)

$$a^2 = a^i a_i. \tag{9a}$$

Thus

$$\left\{\frac{\partial T^{i\alpha}}{\partial F^j_\beta} N_\alpha N_\beta - \rho_0 V^2 \delta^i_j\right\} a'^j = 2\rho_0 V V' a^i$$
$$a_i a'^i = a \tag{11}$$

$$\left\{\frac{\partial T^{i\alpha}}{\partial F^j_\beta} N_\alpha N_\beta - \rho_0 V^2 \delta^i_j\right\} a''^j = 2\rho_0 V V'' a^i + 2\rho_0 (V')^2 a^i + 4\rho_0 V V' a'^i$$
$$+ \frac{\partial^2 T^{i\alpha}}{\partial F^j_\beta \partial F^k_\gamma} N_\alpha N_\beta N_\gamma a'^j a'^k \tag{12}$$
$$a_i a''^i = 1 - a'_i a'^i$$

$$\left\{\frac{\partial T^{i\alpha}}{\partial F^j_\beta} N_\alpha N_\beta - \rho_0 V^2 \delta^i_j\right\} a'''^j = 2\rho_0 V V''' a^i + 6\rho_0 V' V'' a^i + 6\rho_0 (V')^2 a'^i + 6\rho_0 V V'' a'^i$$
$$+ 6\rho_0 V V' a''^i + 3\frac{\partial^2 T^{i\alpha}}{\partial F^j_\beta \partial F^k_\gamma} N_\alpha N_\beta N_\gamma a'^j a''^k \tag{13}$$
$$- \frac{\partial^3 T^{i\alpha}}{\partial F^j_\beta \partial F^k_\gamma \partial F^l_\delta} N_\alpha N_\beta N_\gamma N_\delta a'^j a'^k a'^l$$
$$a_i a'''^i = -3a'_i a''^i.$$

Solution is accomplished as follows. Equations (9) and (9a) are satisfied if $\mathbf{a}(0) = \mathbf{0}$. Then (11) and ($12_2$) are satisfied if $V(0)$ is a characteristic speed, $V(0) = V^{(m)}$, $m = 1, 2, 3$, and $\mathbf{a}'(0)$ is equal to the corresponding unit right proper vector, $\mathbf{a}'(0) = \mathbf{r}^{(m)}$. Since $-\mathbf{r}^{(m)}$ is also a right proper vector, this selection fixes the sign for $a = \pm|\mathbf{a}|$ automatically. Negative roots, $-V^{(m)}$ are assigned to a negative normal, $-\mathbf{N}$. They correspond to waves propagating in the opposite direction.

Let $\mathbf{l}^{(m)}$ be the left proper vectors of the characteristic matrix normalized so that $\mathbf{l}^{(m)} \cdot \mathbf{r}_{(n)} = \delta^{(m)}_{(n)}$. Then the derivative $V'(0)$ is found from (12_1) after multiplication by the appropriate left proper vector. In the special case where stress may be derived from a stored energy function $\mathbf{T} = \partial U/\partial \mathbf{F}$, this leads to the symmetric formula

$$V'(0) = -\frac{1}{4\rho_0 V(0)} \left(\frac{\partial^3 U}{\partial F^i_\alpha \partial F^j_\beta \partial F^k_\gamma}\right)_0 a'^i(0) N_\alpha a'^j(0) N_\beta a'^k(0) N_\gamma. \tag{14}$$

Second derivatives $\mathbf{a}''(0)$ and $V''(0)$ may be found from the two independent equations of (12_1), (this assumes the rank of the characteristic matrix to be two), together with (13_2) and after multiplication by the appropriate left proper vector (13_1). Higher derivatives are found by a similar procedure.

The above discussion suffices to show that if smooth solutions to equation (9) exist, then vanishing amplitude occurs at a branch point. Furthermore, there are three branches, and these correspond exactly to the three possible simple waves in that for each branch the shock speed in the limit of vanishing amplitude is equal to one of the characteristic speeds. However, not all parts of the branches may be accepted as solutions on physical grounds. For reasons of stability it may be argued that a shock wave must travel faster than the corresponding type acceleration wave ahead of the shock and slower than the corresponding type acceleration wave behind the shock. Some such criterion must be substituted for the thermodynamic one of increased entropy across a shock. Thus at least for weak shocks, if $V'(0) > 0$, $[V'(0) < 0]$ then only $a > 0\,(a < 0)$ is admissible, and if $V'(0) = 0$ but $V''(0) > 0\,[V''(0) < 0]$ then any (no) nonzero value of a is admissible. Of course shocks may still be possible for large values of a even if weak shocks are not admissible.

3. CHARACTERISTIC GEOMETRY [12] or [13]

A characteristic surface, $\varphi(\mathbf{X}, t) = 0$ must satisfy the differential equation

$$D(\boldsymbol{\xi}) = \det(C_{ij}^{\alpha\beta}\xi_\alpha\xi_\beta - \rho_0\xi_0^2\delta_{ij}) = 0 \tag{15}$$

where $\boldsymbol{\xi} = (\xi_0, \xi_1, \xi_2, \xi_3) = (\varphi_{,t}, \varphi_{,x}, \varphi_{,y}, \varphi_{,z})$. The four vector $\boldsymbol{\xi}$ is a normal to the surface $\varphi = \text{const.}$ in space-time and the homogeneous function $D(\boldsymbol{\xi})$, when set equal to zero, gives the equation of a cone called the normal cone in $\boldsymbol{\xi}$-space. Equation (15) is bicubic in ξ_0 so that there are three sheets to the cone which may be denoted by

$$\begin{aligned} &\xi_0 = v_N(\xi_\alpha);\ N = 1, 2, 3;\ \alpha = 1, 2, 3 \\ &v_1 \leq v_2 \leq v_3. \end{aligned} \tag{16}$$

If ξ_α is a unit vector, then equation (16) determines the propagation speeds of plane waves with normal ξ_α. If ξ_0 is taken to be a distance along the direction ξ_α then (15) is the equation of the so called velocity surface or reciprocal normal surface. If we set $\xi_0 = -1$ then equation (15) is the intersection of the normal cone with the hyperplane $\xi_0 = -1$ and the surface so determined is called the slowness surface or normal surface. The name slowness is appropriate because the radius in a given direction is the reciprocal of the velocity of plane waves in the same direction. This is clear since the v_N are homogeneous of degree one so that $|\xi_\alpha|^{-1} = |v_N(\xi_\alpha/|\xi_\beta|)|$.

The wave cone is the hypersurface enveloped by the planes whose normals lie in the normal cone. Alternatively it is the hypersurface swept out by the normals to the tangent planes of the normal cone. The generators of the wave cone are tangent at its apex to the bicharacteristic rays which are given by

$$X'_\alpha = A\frac{\partial D}{\partial \xi_\alpha} \qquad \alpha = 0, 1, 2, 3. \tag{17}$$

The prime indicates differentiation with respect to a ray parameter, ρ, and A is an arbitrary proportional factor. The generators of the wave cone are straight lines

$$X_\alpha = A\frac{\partial D}{\partial \xi_\alpha}\rho \qquad \alpha = 0, 1, 2, 3. \tag{18}$$

Since D is homogeneous of degree six in $\boldsymbol{\xi}$ we have $\boldsymbol{\xi} \cdot \mathbf{X} = 6A\rho D = 0$.

The wave surface is the intersection of the wave cone with the hyperplane $t = 1$ so that if ξ_α lies on the slowness surface, points on the wave surface satisfy $X_\alpha \xi_\alpha = 1$, $\alpha = 1, 2, 3$ which may be written

$$X_\alpha N_\alpha = v_N \tag{19}$$

after division by $|\xi_\alpha|$. Thus the wave surface may be regarded as the envelope of all possible plane waves at unit time after passing through the origin. The centers of the various waves and surfaces described above may of course be translated to arbitrary points in space-time. In this paper only two independent space variables are used so that the wave surface, normal surface, etc. reduce to curves in the X–Y plane. It may be shown that the normal curve may have inflection points but has no cusps whereas the wave curve may have cusps but not inflection points. A typical slowness curve and wave curve for an elastic material are shown in Fig. 1.

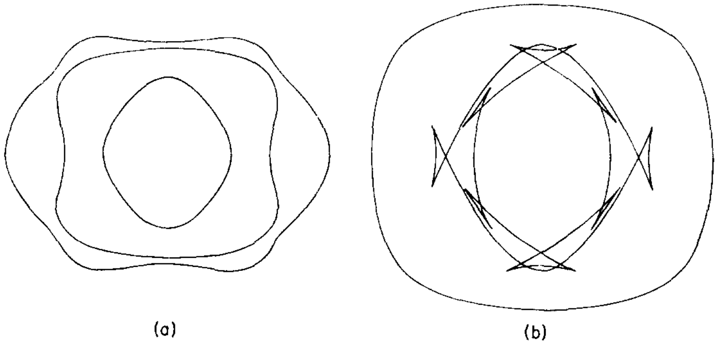

FIG. 1. (a) Slowness curve, and (b) wave curve (after Burridge [14]).

4. REFLECTION OF AN OBLIQUE SHOCK WAVE

Suppose that a plane shock approaches a plane boundary at an oblique angle θ_0 as in Fig. 2. The shock has speed V_s, normal $\mathbf{N}_0 = (\sin\theta_0, -\cos\theta_0, 0)$ amplitude $\mathbf{a}$, deformation gradient and velocity ahead of the wave $\mathbf{F}_{(0)}$, $\mathbf{u}_{(0)}$ and deformation gradient and velocity behind the wave $\mathbf{F}_{(1)}$, $\mathbf{u}_{(1)}$. It is assumed that these quantities satisfy the jump conditions given in equations (2) and (3) and that the shock wave is stable. Thus the shock lies on one of the three branches which satisfy equation (9).

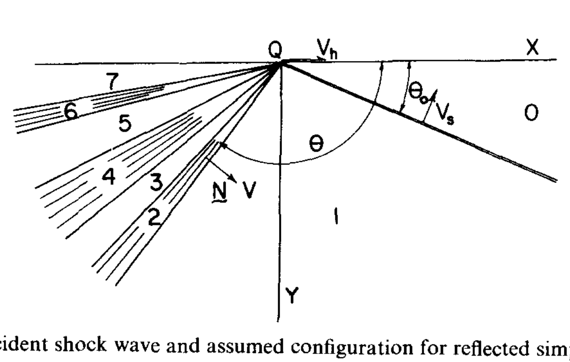

FIG. 2. Incident shock wave and assumed configuration for reflected simple waves.

The reflection problem is to be solved by a semi-inverse approach. It is assumed initially that the reflection consists of three simple waves, one corresponding to each wave sheet. The simple waves are assumed to be separated from the shock, each other, and the boundary by regions of constant velocity and constant deformation gradient. Thus in Fig. 2 regions 2, 4 and 6 are simple waves and regions 0, 1, 3, 5 and 7 have constant state. Within each simple wave every wavelet passes through Q, the point of intersection between the shock wave and the boundary.

Each simple wave is completely described by a one parameter set of functions where the variations of the functions are given by the ordinary differential equation (7). The deformation gradient and velocity are assumed to be continuous throughout regions 1–7. Thus the initial values for the differential equations that describe region 2 are the constant values of region 1, and the final values in region 2 (at the trailing edge of the wave) are the constant values of region 3. In turn region 3 provides the initial values for the wave in region 4 and so on. However, the value of the simple wave parameter in region 2 that maps into the boundary between 2 and 3 is not known *a priori*. Thus the set of initial values for the wave in region 4 is a one parameter family. Similarly the field values in regions 5 and 7 are two and three parameter families respectively, where the three parameters fix the trailing edges of the three simple waves. Boundary conditions at the material surface of region 7 provide in implicit form a set of equations for the three parameters. To complete the problem these equations must be solved for the three simple wave parameters in terms of the specifying parameters of the incident shock wave.

The next three subsections set out in detail the solution described above.

Selection of reflected waves

Here it will be shown that the number and type of reflected waves may be determined by examination of the roots of a certain polynomial. Since the point Q moves along the boundary with speed $V_h = V_s/\sin\theta_0$, the normal speed of a wavelet at angle θ is $V = V_h \sin\theta$ as is clear from Fig. 2. The wavelet normal is given by $\mathbf{N} = (\sin\theta, -\cos\theta, 0)$. For every wavelet in each simple wave choose $L(\gamma) = 0$. Thus equation (8) which specifies the plane of a wavelet may be written

$$0 = X\sin\theta - Y\cos\theta - V_h t\sin\theta. \tag{8'}$$

Let $\tau = \cot\theta$. We have $\tau = (X - V_h t)/Y$. Equation (7) for a simple wave may be written

$$\begin{aligned}(C_{ij}^{\alpha\beta}\hat{N}_\alpha\hat{N}_\beta - \rho_0 V_h^2\delta_{ij})u'_j &= 0\\ V_h F'^j_\beta + u'^j\hat{N}_\beta &= 0\end{aligned} \tag{7'}$$

where the normal divided by $\sin\theta$ has components $\hat{N}_\alpha = (1, -\tau, 0)$. Let $\mathscr{P}(\tau)$ be the polynomial in τ given by

$$\mathscr{P}(\tau) = \det\{C_{ij}^{\alpha\beta}\hat{N}_\alpha\hat{N}_\beta - \rho_0 V_h^2\delta_{ij}\}. \tag{20}$$

The condition that nontrivial solutions for $\mathbf{u}'$ exist is $\mathscr{P}(\tau) = 0$.

In general $\mathscr{P}(\tau)$ is sixth order in τ and in fact can never be less than sixth order. If it were of lower order, then the coefficient of τ^6 would vanish, i.e. $\det\{C_{ij}^{22}\}$. In turn this implies that for the direction $\mathbf{N} = (0, 1, 0)$ one of the characteristic speeds vanishes contrary to assumption.

The real roots of $\mathscr{P}(\tau)$ may be interpreted geometrically as follows. (Refer to Fig. 3.) About a point on the boundary, Q_0, draw all three sheets of the wave curve. Recall that the wave curve is the intersection of the wave cone with the hyperplane $t = 1$ and thus is the envelope over all directions of plane waves which might have passed through Q_0 at $t = 0$. Locate Q on the boundary at a distance V_h from Q_0 and in the direction of passage of the shock. From Q draw all possible tangents to the wave curve. Each tangent determines a real value of $\tau = \cot \theta$, where θ is the angle about Q clockwise from the boundary to the tangent. It is clear that each value of τ so determined is a real root of $\mathscr{P}(\tau)$ and conversely each real root of $\mathscr{P}(\tau)$ corresponds to a line through Q and tangent to the wave curve.†

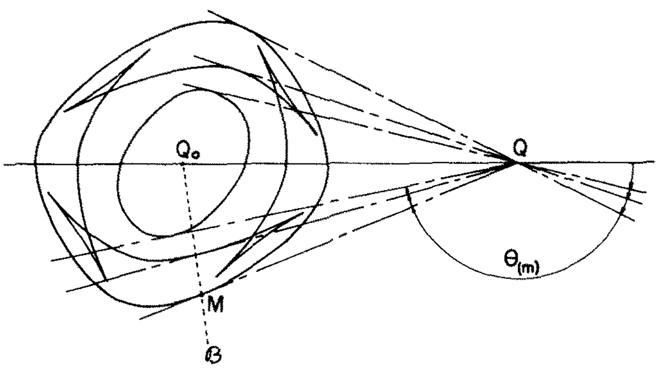

FIG. 3. Geometrical interpretation of the roots of $\mathscr{P}(\tau)$.

From the preceeding geometrical construction and from known properties of the wave curve several statements may be made about the real roots of $\mathscr{P}(\tau)$. In any case, of course, there must be an even number of real roots, i.e. there must be 0, 2, 4 or 6 real roots.

1. If Q lies entirely within the core of a wave sheet (not within a cusped region), then $\mathscr{P}(\tau)$ has no real roots corresponding to that sheet.

2. If Q lies entirely outside a wave sheet, there exist exactly two real roots of $\mathscr{P}(\tau)$ corresponding to that sheet.

3. If Q lies within a singly cusped region of a wave sheet, there exist exactly four real roots of $\mathscr{P}(\tau)$ corresponding to that sheet.

4. If Q lies within a doubly cusped region of a wave sheet, there exist exactly six real roots of $\mathscr{P}(\tau)$ corresponding to that sheet.

5. Simple points on a wave sheet correspond to double roots of $\mathscr{P}(\tau)$ and cusp points correspond to triple roots.

To see properties 1–4 refer to Fig. 4. Consider an arbitrary point M on one of the wave sheets and through the point construct the tangent. The tangent is divided into two parts by M. Let M move around the wave sheet in such a way that the tangent turns continuously clockwise. This is possible since the wave sheets have cusps but not inflection points. As

† The construction given here is dual to the one given by Musgrave for linear anisotropic wave reflections [15] which is based on the slowness curve. The alternate construction could also be used here. The slowness of each wavelet has components $(V_h^{-1}, -\tau V_h^{-1}, 0)$. Values for τ are determined from the intersections with the slowness curve of a line drawn perpendicular to the plane boundary and at a distance V_h^{-1} from the center of the slowness curve. In practice the construction from the slowness curve is probably simpler but the construction from the wave curve is clearer conceptually.

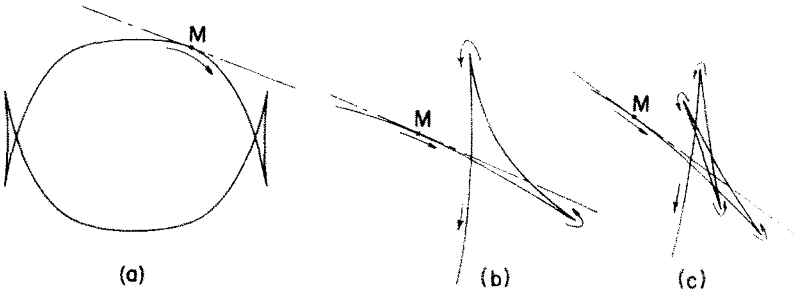

FIG. 4. As M traverses the wave curve, each end of the tangent sweeps over (a) exterior points once, (b) points in a single cusp twice, and (c) points in a double cusp three times. The point M moves according to the arrows.

M moves around the curve it is easy to see that each half of the tangent sweeps over every point (1) of the core interior never, (2) of the exterior once, (3) of the interior of a simple cusp twice, and (4) of the interior of a double cusp three times. The fifth property follows by considering the limit of tangent contact points M as Q approaches a wave sheet.

From this discussion it is clear that if V_h is large enough, which is to say if θ_0 is small enough, then there will exist six real roots of $\mathscr{P}(\tau)$, two for each sheet of the wave curve. All that is required is that Q lie entirely outside all sheets of the wave curve. If Q lies within a cusped region of one of the two inner sheets, there may also be six real roots of $\mathscr{P}(\tau)$. In this last case there seems to be no reason why the solution could not also be carried through formally. However, it may have little significance for the following reason. At the first instant that the angle between the incident shock and the boundary is such that Q lies inside the outer wave sheet, one might expect that a new wave front corresponding to the outer sheet would break away and move ahead of the point Q. If this occurs, nonlinear interaction between the incident shock and the new wave will distort the wave pattern locally so that a solution which involves only plane simple waves will probably no longer be applicable. Such an objection does not carry quite as much weight in the infinitesimal case of course because then there would be no nonlinear interaction or distortion.

The geometrical construction of reflected plane waves may also be viewed in the following way. The moving point Q, at which the shock wave meets the boundary, lies on the line $\mathscr{L}: X = V_h t$ in space-time. In a sense $\mathscr{L}$ is a source for reflected wavelets. Therefore, reflected wavelets should be generated by rays which originate on $\mathscr{L}$ and which by reason of causality point both into the material and forward in time, i.e. toward positive Y and positive t. For $\mathbf{F}$ fixed, the set of all rays through a point on $\mathscr{L}$ generates a wave cone with vertex at the point and the desired reflected wavelets are envelopes of all the wave cones which have vertices on $\mathscr{L}$. The rays which generate the reflected wavelets are precisely those of each cone which lie in the envelope and point toward positive Y and positive t. Let $\mathscr{R}$ be the ray which lies in a reflected wavelet and in the wave cone with vertex at Q_0. Then in the plane $t = 1$ the projected image of $\mathscr{R}$ is a radius of the wave curve about Q_0. The intersection of the reflected wavelet with the plane $t = 1$ is the line which is tangent to the wave curve and which passes through Q, and the radius and tangent meet at the point of tangency to the wave curve. The image of one ray is shown in Fig. 3.

This discussion makes it clear that the only real roots of $\mathscr{P}(\tau)$ which correspond to reflected waves are those for which the tangent point lies in the material since only then will the rays of the wave point into the material. Furthermore, half of the real roots correspond to tangent points in the material and half correspond to tangent points on the other side of the boundary. [In the exceptional case where Q itself lies on the wave curve and $\mathscr{P}(\tau)$ has multiple roots this does not hold of course.] To see this recall that when the point of tangency, M, moves around the sheet, each half of a tangent sweeps through every point in the plane the same number of times as the other half. The point M will lie in the material whenever one half sweeps over Q (the left half for an observer at M facing along $\mathbf{N}$, the direction of propagation) and outside the material when the other half sweeps over Q.

Thus there are at most three real roots of $\mathscr{P}(\tau)$ that correspond to reflected waves. If Q lies outside all three wave sheets, the three algebraically smallest values of τ correspond to reflected waves. This is the most important case and the one of primary concern in this paper.

In addition to the requirement $\mathscr{P}(\tau) = 0$, in order to satisfy ($7'_1$) it must be required that $\mathbf{u}'$ be a right proper vector of $Q_{ij} = C_{ij}^{\alpha\beta} N_\alpha N_\beta$. A distinct root of $\mathscr{P}(\tau)$ determines a fixed direction $\mathbf{N}$ and for each fixed direction it has been assumed that $\mathbf{Q}$ has three distinct proper numbers and hence three unique (within a scalar) right proper vectors. Thus $\mathbf{u}'$ may be written

$$\mathbf{u}' = V_h w(\mathbf{F})\mathbf{r}(\mathbf{F}) \tag{21}$$

where w is an arbitrary scaler function and $\mathbf{r}(\mathbf{F})$ is any right proper vector that corresponds to direction $\mathbf{N}$ and the proper number $\rho_0 V_h^2 \sin^2\theta$.

Distribution of wavelets

To complete the reflection problem it remains only to determine the strength and angle of every wavelet within the three simple waves where each simple wave is described by a system of ordinary differential equations. Let the three roots of $\mathscr{P}(\tau)$ that correspond to reflected waves be denoted by $\tau = \sigma_{(m)}(\mathbf{F})$ where $m = 2, 4$ or 6 and

$$\sigma_{(2)}(\mathbf{F}) > \sigma_{(4)}(\mathbf{F}) > \sigma_{(6)}(\mathbf{F}).$$

The appropriate equations follow from ($7'_2$) and (21)

$$\begin{aligned} F'^{j}_{(m)1} &= -w_{(m)}(\mathbf{F})r^j_{(m)}(\mathbf{F}) \\ F'^{j}_{(m)2} &= w_{(m)}(\mathbf{F})\sigma_{(m)}(\mathbf{F})r^j_{(m)}(\mathbf{F}) \\ F'^{j}_{(m)3} &= 0 \\ u'^{j}_{(m)} &= V_h w_{(m)}(\mathbf{F})r^j_{(m)}(\mathbf{F}). \end{aligned} \tag{22}$$

Equations (22) are subject to the initial conditions

$$\begin{aligned} \mathbf{F}^{(0)}_{(m)} &= \mathbf{F}_{(m-1)} \\ \mathbf{u}^{(0)}_{(m)} &= \mathbf{u}_{(m-1)} \end{aligned} \tag{23}$$

that is to say, the field values in the simple waves are continuous across the wave fronts. The functions $w_{(m)}(\mathbf{F})$ may be chosen for convenience since they represent only a relabeling of the wavelets in the sense that the systems

$$\frac{\mathrm{d}y}{\mathrm{d}\gamma} = f(y) \quad \text{and} \quad \frac{\mathrm{d}y}{\mathrm{d}\hat{\gamma}} = w(y)f(y), \qquad \frac{\mathrm{d}\gamma}{\mathrm{d}\hat{\gamma}} = w(y)$$

are equivalent. They are strictly equivalent if w has no zeros but since the label γ or $\hat{\gamma}$ has no intrinsic importance, the function $w(\mathbf{F})$ may actually be chosen so as to ensure uniqueness or ease of solution or for any other desirable property. Of course once the solution $\mathbf{F}(\gamma)$ is found, the distribution of the wavelets follows by substitution so that $\tau = \sigma[\mathbf{F}(\gamma)]$. Clearly $F^j_{(m)3} = F^j_{(1)3}$ for each m and the solution of (22_4) follows by quadrature once the solution to (22_1) and (22_2) is known.

The value of $\mathbf{F}$ changes through a simple wave, but the ordering of the values of $\sigma^{(2)}$, $\sigma^{(4)}$ and $\sigma^{(6)}$ cannot change. If they could change, then, since the roots of a polynomial of a fixed order are continuous functions of the polynomial coefficients which for the problem at hand in turn are continuous functions of $\mathbf{F}$, and hence of the wave parameter, γ, then for some value of γ, two roots of $\mathscr{P}(\tau)$ would be equal. This implies that for a certain direction of propagation two characteristic speeds are equal, but this case has been ruled out by assumption. The ordering ensures that the three simple waves will in fact be separated by regions of constant state.

Now let λ, μ and ν be the independent parameters in waves (2), (4) and (6) respectively and define the vector $\hat{\mathbf{N}}^{(m)}(\mathbf{F}) = [1, -\sigma^{(m)}(\mathbf{F}), 0]$. We have (refer to Fig. 2)

(i) In region (1)

$$\mathbf{F} = \mathbf{F}_{(1)}. \tag{24}$$

(ii) In region (2)

$$\left.\begin{aligned} \mathbf{F} &= \mathbf{F}_{(2)}(\lambda) \\ \tau &= \sigma^{(2)}(\mathbf{F}) \end{aligned}\right\} \quad \text{on } 0 \leq \lambda \leq \tilde{\lambda} \tag{25}$$

where

$$\begin{aligned} \frac{\mathrm{d}F^i_{(2)\alpha}}{\mathrm{d}\lambda} &= -\hat{N}^{(2)}_\alpha(\mathbf{F}) r^i_{(2)}(\mathbf{F}) w_{(2)}(\mathbf{F}) \\ \mathbf{F}_{(2)}(0) &= \mathbf{F}_{(1)}. \end{aligned} \tag{26}$$

(iii) In region (3)

$$\mathbf{F} = \mathbf{F}_{(3)} = \mathbf{F}_{(2)}(\tilde{\lambda}). \tag{27}$$

(iv) In region (4)

$$\left.\begin{aligned} \mathbf{F} &= \mathbf{F}_{(4)}(\mu;\tilde{\lambda}) \\ \tau &= \sigma^{(4)}(\mathbf{F}) \end{aligned}\right\} \quad \text{on } 0 \leq \mu \leq \tilde{\mu} \tag{28}$$

where

$$\begin{aligned} \frac{\mathrm{d}F^i_{(4)\alpha}}{\mathrm{d}\mu} &= -\hat{N}^{(4)}_\alpha(\mathbf{F}) r^i_{(4)}(\mathbf{F}) w_{(4)}(\mathbf{F}) \\ \mathbf{F}_{(4)}(0) &= \mathbf{F}_{(3)}. \end{aligned} \tag{29}$$

(v) In region (5)

$$\mathbf{F} = \mathbf{F}_{(5)} = \mathbf{F}_{(4)}(\tilde{\mu};\tilde{\lambda}). \tag{30}$$

(vi) In region (6)

$$\left.\begin{aligned}\mathbf{F} &= \mathbf{F}_{(6)}(v;\tilde{\mu},\tilde{\lambda})\\ \tau &= \sigma_{(6)}(\mathbf{F})\end{aligned}\right\}\quad \text{on } 0 \leq v \leq \tilde{v} \tag{31}$$

where

$$\frac{\mathrm{d}F^i_{(6)\alpha}}{\mathrm{d}v} = -\hat{N}^{(6)}_\alpha(\mathbf{F})r^i_{(6)}(\mathbf{F})w_{(6)}(\mathbf{F}) \tag{32}$$

$$\mathbf{F}_{(6)}(0) = \mathbf{F}_{(5)}.$$

(vii) In region 7

$$\mathbf{F} = \mathbf{F}_{(7)} = F_{(6)}(\tilde{v};\tilde{\mu},\tilde{\lambda}). \tag{33}$$

To complete the solution a boundary condition for region (7) must be met. The extreme conditions are the completely hard or clamped boundary $\mathbf{u} = 0$ and the completely soft or free boundary $T^{i\alpha}N_\alpha = 0$ where $\mathbf{N} = (0, -1, 0)$. In the general case there will be three boundary conditions from which to calculate the three parameters $\tilde{\lambda}$, $\tilde{\mu}$, $\tilde{v}$ in terms of the deformation gradient ahead of the shock, $\mathbf{F}_{(0)}$, incident angle, θ_0, shock type and amplitude, a.

Comments on existence

No general existence theory exists for the solution of the mathematical problem posed by equations (24)–(33) together with an appropriate boundary condition. A complete theory would no doubt depend strongly on the properties of the constitutive functions as well as the parameters of the incident shock wave.

However, it is possible to state a simple existence theorem for small but finite amplitude waves. If for a fixed $\mathbf{F}_{(0)}$, θ_0 and incident shock wave type, there is a unique reflection solution for the infinitesimal theory linearized about the deformation gradient $\mathbf{F}_{(0)}$ and velocity $\mathbf{u}_{(0)}$ then there is a neighborhood about $\tilde{\lambda} = \tilde{\mu} = \tilde{v} = 0$ and $a = 0$ (i.e. vanishing amplitude of the incident shock) within which the finite reflection problem has a unique solution for $\tilde{\lambda}$, $\tilde{\mu}$ and $\tilde{v}$ in terms of a and the parameters $\mathbf{F}_{(0)}$ and θ_0. This follows from a simple application of the implicit function theorem. For fixed $\mathbf{F}_{(0)}$ and θ_0 the values $\tilde{\lambda} = \tilde{\mu} = \tilde{v} = a = 0$ satisfy the boundary conditions and in each simple wave the deformation gradient and velocity take on only their initial value. That is to say, the case of vanishing shock amplitude has the solution of vanishing reflections. Let the boundary conditions to be met be denoted by $B^i(\tilde{\lambda}, \tilde{\mu}, \tilde{v}, a; \mathbf{F}_{(0)}, \theta_0) = 0$† and let $J(\tilde{\lambda}, \tilde{\mu}, \tilde{v}, a; \mathbf{F}_{(0)}, \theta_0)$ denote the Jacobian of the derivatives of B^i taken with respect to $\tilde{\lambda}$, $\tilde{\mu}$, $\tilde{v}$. Then the condition $J(0, 0, 0, 0; \mathbf{F}_{(0)}, \theta_0) \neq 0$ is simultaneously the condition for invertibility of $B^i = 0$ in a neighborhood of $\tilde{\lambda} = \tilde{\mu} = \tilde{v} = a = 0$ and the condition for solution of the linear problem as may easily be verified at least for the clamped or free boundary.

Once $\tilde{\lambda}$, $\tilde{\mu}$ and $\tilde{v}$ have been determined, the geometry and distribution of wavelets are completely determined and the results may be accepted as a solution to the reflection problem provided that within each simple wave the function $\sigma^{(m)}(\mathbf{F})$ is a real monotonically decreasing function of the appropriate parameters. This corresponds to monotonically increasing angle θ as the parameter ranges from zero to its maximum value (or minimum

† For example $T^{i2}(\mathbf{F}_{(7)}) = 0$, free boundary; or $u^i_{(7)} = 0$ clamped boundary.

value if say $\tilde{\lambda} < 0$) and is required to insure a single valued solution of the reflection problem as a whole.

Failure of the simple wave solution and the possibility of reflected shocks

It may happen that for one or more of the simple waves the function $\sigma^{(m)}(\mathbf{F})$ is not monotonically decreasing over the whole range of the parameter. If it increases monotonically, then the reflected simple wave should probably be replaced by a reflected shock wave of the same type. In this case it is necessary to solve equations (9) and (9a) for θ and $\mathbf{a}$ as functions of amplitude a on the appropriate branch. In these equations $V = V_h \sin\theta$, $\mathbf{N} = (\sin\theta, -\cos\theta, 0)$ and $\mathbf{F}_{(+)}$ is the deformation gradient ahead of the shock. Once again a reflected wave may be specified completely in terms of a single parameter so that in region (7) there are again three parameters to be determined from the boundary conditions. All shock waves in the solution must be checked for stability of course. Finally, it should be noted that as before, if the linear problem has a solution, then for this case as well there is a solution in the neighborhood of vanishing incident shock and vanishing reflected waves. The proof is the same as before.

Another possibility is that $\sigma^{(m)}(\mathbf{F})$ for one or more of the waves is real but is not monotonic at all. In this case the simple wave should probably be replaced by a combination shock and simple wave. This case is more complicated than the previous ones and will not be discussed further.

Comments on critical angles

The solution by simple waves fails if one of the functions $\sigma^{(m)}(\mathbf{F})$ becomes complex for some value of the parameter. Recall that complex values of $\sigma^{(m)}(\mathbf{F})$ correspond to the point Q lying within the mth sheet of the wave curve. When Q lies on the outer wave sheet, the critical angle has been reached for simple wave reflections. This phenomenon is amplitude dependent as well as orientation dependent. For fixed amplitude of the incident shock the point Q moves toward the outer wave sheet as the incident angle θ_0 increases. This is similar to the linear case except that here the wave sheet geometry for the leading edge of the reflected simple wave depends on $\mathbf{F}_{(1)}$ which in turn depends on θ_0 and the amplitude of the incident shock. In order that the shock wave be stable it must intersect the corresponding wave sheet if both are extended through the material boundary (see Fig. 5). This is

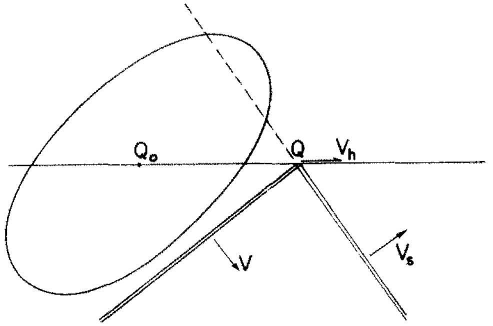

FIG. 5. Relation between incident and reflected shock waves and wave curves.

required since acceleration waves of the same type behind the shock must travel faster than the shock itself. Thus even for fixed angles of incidence the point Q may approach the outer wave sheet as amplitude increases.

This discussion should make it clear that critical angles occur for shock waves that correspond to the outer wave sheet as well as for those that correspond to the inner sheets. This is in direct contrast to linear theory where critical angles occur only for the inner sheets.

Similarly if the first reflected wave is a shock wave it may happen that for some incident angles or amplitudes the solution fails. Whereas the leading edge of a simple wave at angle θ must be tangent to the outer wave sheet computed for $\mathbf{F}_{(1)}$ a shock wave must lie at an angle less than θ since for reasons of stability it must have a higher speed of propagation than that of an acceleration wave of the same type in the region ahead of the shock (see Fig. 5). Furthermore, the exact angle of the reflection depends on the strength of the reflection as well. In any case if the solution fails it fails for angles of incidence and amplitudes such that the point Q lies outside the wave sheet. Thus for a fixed amplitude of the incident shock, the critical angle is smaller for the case of reflected shock waves than for the case of reflected simple waves.

If a critical angle occurs for either the reflected simple wave or reflected shock wave, it is for the same reason physically. The point at which the plane shock wave would intersect the material surface does not move fast enough to keep up with the slowest plane waves which are consistent with the outer wave sheet. Consequently waves break away from point Q and interact with and modify the incident shock itself. There are two possibilities. On the one hand the incident shock may be strengthened and tend to speed up along the boundary and in this case it seems likely that a mach stem will form. On the other hand the incident shock may be weakened and tend to slow down along the boundary and in that case the shock may become curved.

5. DISCUSSION OF SPECIAL CASES

Transverse symmetry

Three reflected waves are needed generally to meet boundary conditions. Nevertheless, if the material and the deformation possess sufficient symmetry with respect to the X–Y plane, only two reflected waves may be required. Sufficient conditions for this to occur are as follows. Direct tensor notation is used.

According to the rules of material symmetry and the principle of material indifference [9] the following holds for arbitrary rotations $\mathbf{R}$ and for every unimodular tensor $\mathbf{H}$ where $\mathbf{H}$ belongs to the isotropy group of the material.

$$\mathbf{T}(\mathbf{F}) = (\det \mathbf{R})^{-1}(\det \mathbf{H})^{-1}\mathbf{R}^T\mathbf{T}(\mathbf{RFH})\mathbf{H}^T. \tag{34}$$

Now suppose that in the given Cartesian frame $\mathbf{F}$ has the representation

$$\|\mathbf{F}\| = \begin{Vmatrix} l & m & n \\ p & q & r \\ s & t & u \end{Vmatrix}. \tag{35}$$

If particular **R** and **H** both have the representation

$$\begin{Vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{Vmatrix}$$

then equation (34) implies that

$$\begin{aligned} &\begin{Vmatrix} T^{11} & T^{12} & T^{13} \\ T^{21} & T^{22} & T^{23} \\ T^{31} & T^{32} & T^{33} \end{Vmatrix} \text{ as functions of } \begin{Vmatrix} l & m & n \\ p & q & r \\ s & t & u \end{Vmatrix} \\ &= \begin{Vmatrix} T^{11} & T^{12} & -T^{13} \\ T^{21} & T^{22} & -T^{23} \\ -T^{31} & -T^{32} & T^{33} \end{Vmatrix} \text{ as functions of } \begin{Vmatrix} l & m & -n \\ p & q & -r \\ -s & -t & u \end{Vmatrix}. \end{aligned} \tag{36}$$

Suppose $n = r = t = 0$. Then T^{11}, T^{12}, T^{21}, T^{22} are even in s, and $\partial T^{11}/\partial s$ etc. are odd in s and hence vanish for $s = 0$. Furthermore T^{31}, T^{32}, T^{13}, T^{23} and all their derivatives with respect to l are odd in s and hence vanish for $s = 0$. These and similar arguments concerning the behavior of **T** as a function of the other components of **F** demonstrate that for all normals $\mathbf{N} = (\alpha, \beta, 0)$ and deformations

$$\|\mathbf{F}\| = \begin{Vmatrix} l & m & 0 \\ p & q & 0 \\ 0 & 0 & u \end{Vmatrix} \tag{37}$$

the acoustic tensor **Q** has the representation

$$\|\mathbf{Q}\| = \begin{Vmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & e \end{Vmatrix}. \tag{38}$$

The tensor **Q** has one right proper vector along the z axis and two in the x–y plane. This fact together with equation (7_2) implies that for two reflected waves $F'^{j}_{\alpha} = 0$ if either $j = 3$ or $\alpha = 3$. Thus if **F** in the region ahead of these two waves is given by equation (37) then it has the same representation in each wave and in the adjacent constant regions. Similarly, since $\mathbf{u}'$ is proportional to the right proper vectors of **Q**, if **u** lies in the x–y plane ahead of the two waves, then it also lies in the x–y plane in the waves and in the adjacent constant regions. Thus throughout the material $T^{32} = 0$ and $u^3 = 0$. Examination of equations (11_1), (12_1) and (13_1) indicates that if **F** has the form of equation (37) ahead of a shock wave then there are two waves possible with amplitude vector **a** lying in the x–y plane and **F** retains the form of equation (37) behind the wave. This is true since $\mathbf{a}'(0)$ is a right proper vector of **Q** and it lies in the x–y plane. Thus all higher derivatives $\mathbf{a}''(0)$, $\mathbf{a}'''(0)$, etc. lie in the x–y plane and $\mathbf{a}(a)$ expanded in a power series about $a = 0$ lies in the x–y plane.

The preceding arguments may be summarized by the following statement. If a shock wave with polarization in the x–y plane propagates into a region with deformation given by equation (37), if the material is symmetric with respect to reflections across the X–Y plane, and if one boundary condition to be met is either $T^{32} = 0$ or $u^3 = 0$ on $Y = 0$ then at most two reflected waves are required. In particular the statement covers the important case of reflections in an isotropic material in which a normal shock or a quasi-transverse shock with polarization in the x–y plane propagates into a region in which one boundary condition is as above.† However, it does not cover the case of a quasi-transverse shock with a non-zero z-component of polarization.

Incompressible materials

In an incompressible elastic material there are only two sheets to the wave cone rather than three and both waves are always of strictly transverse type. If a reflection solution in terms of simple waves is attempted, the deformation and velocity behind the wave will be given in terms of two parameters, yet three boundary conditions must be met. In general the problem will have no solution for in effect the third sheet has infinite propagation speed and every case is a critical angle case except for normal incidence. However, solutions may exist for special cases in which the material and the deformation exhibit degeneracy so that there exists some functional dependence among the boundary conditions.

As an example of this degenerate type behavior consider an incompressible isotropic material with constitutive equation

$$\mathbf{T} = -p(\mathbf{F}^{-1})^T + h_1(\mathrm{I}_\mathbf{B}, \mathrm{II}_\mathbf{B})\mathbf{F}. \tag{39}$$

Here p is an arbitrary function and h_1 is a smooth function of the first two invariants of the left Cauchy–Green strain tensor $\mathbf{B} = \mathbf{F}\mathbf{F}^T$. This is a modest generalization of Rivlin's neo-Hookean material [17] in which h_1 is a constant.‡ Further assume the deformation gradient to have the form

$$\|\mathbf{F}\| = \left\| \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ z_{,X} & z_{,Y} & 1 \end{matrix} \right\|, \quad \|(\mathbf{F}^{-1})^T\| = \left\| \begin{matrix} 1 & 0 & -z_{,X} \\ 0 & 1 & -z_{,Y} \\ 0 & 0 & 1 \end{matrix} \right\| \tag{40}$$

and that $\dot{x} = \dot{y} = 0$. All motion is restricted to the z direction. Thus there is degeneracy in both the constitutive relation and in the deformations allowed. The invariants are given by

$$\begin{aligned} \mathrm{I}_\mathbf{B} &= 1 + z_{,X}^2 + z_{,Y}^2 \\ \mathrm{II}_\mathbf{B} &= -(1 + z_{,X}^2 + z_{,Y}^2). \end{aligned} \tag{41}$$

It is convenient for the restricted deformation assumed here to let

$$h_1(\mathrm{I}_\mathbf{B}, \mathrm{II}_\mathbf{B}) = h[\tfrac{1}{2}(z_{,X}^2 + z_{,Y}^2)] \tag{42}$$

where h is a function of only one argument. If we set $p = h$ then the momentum equation (1_1) is satisfied for $i = 1, 2$. Application of the theory of simple waves to the third equation alone leads to

$$\begin{aligned} (C^{\alpha\beta}N_\alpha N_\beta - \rho_0 V^2)\dot{z}' &= 0 \\ Vz'_{,\alpha} + N_\alpha \dot{z}' &= 0 \end{aligned} \tag{43}$$

† The slowness curves and wave curves appropriate for these cases have recently been described in detail by Payton [16].

‡ The most general incompressible isotropic material has constitutive equation [9] $\mathbf{T} = -\mathscr{P}(\mathbf{F}^{-1})^T + h_1(\mathrm{I}_\mathbf{B}, \mathrm{II}_\mathbf{B})\mathbf{F} + h_2(\mathrm{I}_\mathbf{B}, \mathrm{II}_\mathbf{B})\mathbf{F}\mathbf{F}^T\mathbf{F}$.

where $\alpha, \beta = 1, 2$ and

$$C^{\alpha\beta} = \frac{\partial T^{3\alpha}}{\partial z_{,\beta}} = \delta_{\alpha\beta}h + z_{,\alpha}z_{,\beta}`h. \tag{44}$$

The backward prime indicates differentiation with respect to the argument of h whereas the ordinary prime indicates differentiation with respect to the simple wave parameter as before. Wave speeds are given by

$$V = \pm\{\rho_0^{-1}[h+(z_{,\alpha}N_\alpha)^2`h]\}^{\frac{1}{2}} \tag{45}$$

and $\dot{z}'$ is arbitrary. Resolved shear traction on planes $Z =$ const. is given by

$$T = [(T^{13})^2+(T^{23})^2]^{\frac{1}{2}} = vh(\tfrac{1}{2}v^2) \tag{46}$$

where $v = (z_{,\alpha}z_{,\alpha})^{\frac{1}{2}}$. The reciprocal normal curve or velocity curve, given by equation (45) takes on its extreme values when $\mathbf{N}$ is parallel or orthogonal to ∇z. These extreme values are T/v and $\mathrm{d}T/\mathrm{d}v$. If $\mathrm{d}T/\mathrm{d}v > 0$ for all v, then wave speeds are always real since T vanishes when v does.

Suppose a shock of amplitude a propagates into an unstressed region. Then for arbitrary angle with the boundary, θ_0, the shock speed is given by

$$V_s = \left[\frac{T(a)/a}{\rho_0}\right]^{\frac{1}{2}} = \left[\frac{h(\frac{1}{2}a^2)}{\rho_0}\right]^{\frac{1}{2}} \tag{47}$$

where behind the shock $v = a$, $z_{,\alpha} = -aN_\alpha$, $\mathbf{N} = (\sin\theta_0, -\cos\theta_0, 0)$ and $\dot{z} = aV_s$. Assume $T(v)$ to be concave up so that the shock is stable. Reflected wavelets lie on planes with cotangent given by

$$\tau = \frac{z_{,X}z_{,Y}`h}{h+z_{,Y}^2`h} - \left\{\frac{\rho_0 V_h^2}{h+z_{,Y}^2`h} - \frac{h(h+v^2`h)}{(h+z_{,Y}^2`h)^2}\right\}^{\frac{1}{2}}. \tag{48}$$

For τ to be real the term in brackets must be positive. For the case when the reflected wave is a simple wave, to be less than the critical angle the incident angle must satisfy

$$\cot^2\theta_0 > \frac{a^2`h(a)}{h(a)+a^2`h(a)} \tag{49}$$

To complete the problem the following equations must be integrated.

$$\begin{aligned} \dot{z}' &= V_h f; & \dot{z}(0) &= aV_s \\ z'_{,X} &= -f & z_{,X}(0) &= -a\sin\theta_0 \\ z'_{,Y} &= \tau f; & z_{,Y}(0) &= a\cos\theta_0 \end{aligned} \tag{50}$$

where f is an arbitrary function.

If the boundary condition to be met is $T^{32} = z_{,Y}h = 0$, it is convenient to choose $f = -\tau^{-1}$ for then

$$\begin{aligned} z_{,Y}(\lambda) &= -\lambda + z_{,Y}(0) \\ \dot{z}(\lambda) &= -V_h[z_{,X}(\lambda) - z_{,X}(0)] + \dot{z}(0) \end{aligned} \tag{51}$$

where $dz_{,X}/d\lambda = \tau^{-1}$, $z_{,X}(0) = -a\sin\theta_0$ and the boundary condition is satisfied when $\lambda = z_{,Y}(0)$. Substitution for $z_{,X}$ and $z_{,Y}$ in the equation for τ completes the solution. If the boundary condition to be met is $\dot{z} = 0$ on $Y = 0$, it is convenient to choose $f = -V_h^{-1}$ for then $\dot{z}(\lambda) = -\lambda + \dot{z}(0)$ and the boundary condition is satisfied when $\lambda = \dot{z}(0)$.

Elastic fluid

Another example, more realistic than the last but still a rather degenerate case is that of an elastic fluid. In this case the only stress is a normal pressure, p, which depends only on the density. The Piola stress tensor is

$$\mathbf{T} = -Jp(J)(\mathbf{F}^{-1})^T \tag{52}$$

where $J = \det\mathbf{F}$. The elasticities are given by

$$C_{ij}^{\alpha\beta} = -J(p + J\,{}'p)X^{\alpha}_{,i}X^{\beta}_{,j} + JpX^{\alpha}_{,j}X^{\beta}_{,i} \tag{53}$$

Here again the backward prime indicates differentiation with respect to the argument. Velocity of propagation is given by

$$V = \pm[-\rho_0^{-1}\,{}'pJ^2X^{\alpha}_{,i}X^{\beta}_{,i}N_\alpha N_\beta]^{\frac{1}{2}} \tag{54}$$

For V to be real we must have ${}'p(J) < 0$. Equation (54) is equivalent to the more usual expression

$$v = \pm\sqrt{\left(\frac{dp}{d\rho}\right)} - \mathbf{n}\,.\,\mathbf{u} \tag{55}$$

where v is the spatial velocity of the wave, $\mathbf{n}$ is the spatial normal and ρ is the density $\rho = \rho_0 J^{-1}$. The derivative of the particle velocity is given by

$$u'^j = wX^{\alpha}_{,j}N_\alpha \tag{56}$$

where w is an arbitrary function and $n_j = kX^{\alpha}_{,j}N_\alpha$, that is to say $\mathbf{u}'$ is parallel to the spatial normal, $\mathbf{n}$.

Suppose a shock of amplitude a propagates into an unstressed region. Then for arbitrary angle with the boundary, θ_0, the shock speed is given by

$$V_s = \left[\frac{p(1-a) - p(1)}{\rho_0 a}\right]^{\frac{1}{2}} \tag{57}$$

and behind the shock $u^i = aV_s n_i$ and $x^i_{,\alpha} = \delta^i_\alpha - an_iN_\alpha$. The material and spatial normals are equal $\mathbf{N}_0 = \mathbf{n}_0 = (\sin\theta_0, -\cos\theta_0, 0)$. Assume $p(\rho)$ to be concave up so that the shock is stable.

The normals in each reflected wavelet are $\mathbf{N} = (\sin\theta, -\cos\theta, 0)$ so that by equation (7_2) $x'^i_{,3} = 0$, which together with the appropriate initial conditions implies that $x^i_{,3} = \delta^i_3$ throughout the reflection zone. This in turn implies that $X^\alpha_{,3} = \delta^\alpha_3$ so that by equation (56) $u'^3 = 0$ and by equation (7_2) $x'^3_{,\alpha} = 0$. We have $x^3_{,\alpha} = \delta^3_\alpha$. Thus there are only four components of the deformation gradient to be found.

$$\|\mathbf{F}\| = \left\|\begin{matrix} x_{,X} & x_{,Y} & 0 \\ y_{,X} & y_{,Y} & 0 \\ 0 & 0 & 1 \end{matrix}\right\|, \quad \|\mathbf{F}^{-1}\| = 1/J \left\|\begin{matrix} y_{,Y} & -x_{,Y} & 0 \\ -y_{,X} & x_{,X} & 0 \\ 0 & 0 & J \end{matrix}\right\| \tag{58}$$

$$J = x_{,X}y_{,Y} - x_{,Y}y_{,X}.$$

The cotangent of the angle for each wavelet is given by

$$\tau = -\frac{y_{,Y}y_{,X}+x_{,Y}x_{,X}}{y_{,X}^2+x_{,X}^2} - \frac{J}{y_{,X}^2+x_{,X}^2}\left\{\frac{y_{,X}^2+x_{,X}^2}{(-\wp' J^2/\rho_0 V_h^2)}-1\right\}^{\frac{1}{2}}. \tag{59}$$

For real τ the term in brackets must be non-negative. In the case of a reflected simple wave this requirement again leads to an expression which the incident angle must satisfy in order not to equal or exceed the critical angle.

$$\cot^2\theta_0 > \frac{c^2}{V_s^2}-(1-a)^2 \tag{60}$$

where c is the acoustic speed behind the shock $c^2 = \mathrm{d}\wp/\mathrm{d}\rho$ evaluated at $\rho = \rho_0(1-a)^{-1}$.

To complete the solution the following equations must be integrated for $i = 1, 2$ and $\alpha = 1, 2$

$$\begin{aligned} u'^i &= wX^\alpha_{,i}\hat{N}_\alpha; & u^i(0) &= aV_s n_i(0) \\ V_h x'^i_{,\alpha} &= -wX^\beta_{,i}\hat{N}_\beta\hat{N}_\alpha; & x^i_{,\alpha}(0) &= \delta^i_\alpha - an_i(0)N_\alpha(0). \end{aligned} \tag{61}$$

In these equations w is an arbitrary function of $x^i_{,\alpha}$, $X^\alpha_{,i}$ is given by (58_2), $\hat{\mathbf{N}} = (1, -\tau, 0)$, τ is given by (59), $\mathbf{n}(0) = \mathbf{N}(0) = (\sin\theta_0, -\cos\theta_0, 0)$, $V_h = V_s/\sin\theta_0$ and V_s is given by (57). If the boundary condition is given in terms of pressure, so that the density and Jacobian are known for $Y = 0$, it is convenient to choose $w = -V_h(X^\alpha_{,i}X^\beta_{,i}\hat{\mathrm{N}}_\alpha\hat{\mathrm{N}}_\beta)^{-1} = -V_h^{-1}\,\mathrm{d}\wp/\mathrm{d}\rho$. In this case the simple wave parameter is related to J by $\lambda = \ln(J/1-a)$ as may easily be seen by multiplying (61_2) by $JX^\alpha_{,i}$ and substituting for w so that the equation becomes $J' = J$.

Other problems

The methods used to solve the oblique shock reflection problem may also be used to solve a number of closely related problems. For example, in the case of a plane shock wave which impinges at an oblique angle on a plane of material discontinuity, it is natural to assume the presence of three reflected simple waves or shocks and three transmitted simple waves or shocks. Each shock and each simple wavelet must pass through the same point, Q, which moves along the common boundary at constant speed. There will be six wave parameters to be determined from the six conditions that velocity and traction be continuous at the plane boundary. A critical angle may occur for material on either side of the boundary.

As another example consider the case where the boundary condition prescribes a moving step discontinuity in surface traction or surface velocity. The discontinuity is to move at a constant speed V_h along the boundary and separates two semi-infinite regions of constant surface traction or velocity. The case of discontinuous velocity is a generalization of the well known expansion or contraction corner in steady gas dynamics. In either case three simple waves or shocks trail from the moving point of discontinuity.

Finally it should be pointed out that the case of normal reflection or normal wave generation by step loading can be treated in a way virtually identical to the oblique reflection case with the additional simplification that $\mathbf{N} = (0, 1, 0)$ for all three waves. There is no need to find τ as a function of deformation gradient and equations (7) may be used directly.

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(*Received* 19 *December* 1969; *revised* 16 *April* 1970)

Абстракт—Исследуется отражение конечной упругой, плоской, ударной волны от плоской перегородки. Используется полуобратный метод решения. Рассматриваются подробно только углы удара меньше критического угла.

Вообще, существуют три отраженные волны, которые могут быть простыми волнами или ударными.

Находится, что каждый род отраженной волны можно полно определить одинарным параметром, если условия на фронте волны известны. Граничные условия на плоской поверхности должны быть решены для определенных параметров. Формулируется простая теорема существования; обсуждаются случаи критического угла и даются некоторые простые примери.